

On separability of the functional space with the open-point and bi-point-open topologies, II

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Abstract

In this paper we continue to study the property of separability of functional space $C(X)$ with the open-point and bi-point-open topologies. We show that for every perfect Polish space X a set $C(X)$ with bi-point-open topology is a separable space. We also show in a set model (the iterated perfect set model) that for every regular space X with countable network a set $C(X)$ with bi-point-open topology is a separable space if and only if a dispersion character $\Delta(X) = \mathfrak{c}$.

Keywords: open-point topology, bi-point-open topology, separability
2000 MSC: 54C40, 54C35, 54D60, 54H11, 46E10

1. Introduction

The space $C(X)$ with the topology of pointwise convergence is denoted by $C_p(X)$. It has a subbase consisting of sets of the form

$$[x, U]^+ = \{f \in C(X) : f(x) \in U\},$$

where $x \in X$ and U is an open subset of real line \mathbb{R} .

In paper [3] was introduced two new topologies on $C(X)$ that we call the open-point topology and the bi-point-open topology. The open-point topology on $C(X)$ has a subbase consisting of sets of the form

$$[V, r]^- = \{f \in C(X) : f^{-1}(r) \cap V \neq \emptyset\},$$

where V is an open subset of X and $r \in \mathbb{R}$. The open-point topology on $C(X)$ is denoted by h and the space $C(X)$ equipped with the open-point topology h is denoted by $C_h(X)$.

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Now the bi-point-open topology on $C(X)$ is the join of the point-open topology p and the open-point topology h . It is the topology having subbase open sets of both kind: $[x, U]^+$ and $[V, r]^-$, where $x \in X$ and U is an open subset of \mathbb{R} , while V is an open subset of X and $r \in \mathbb{R}$. The bi-point-open topology on the space $C(X)$ is denoted by ph and the space $C(X)$ equipped with the bi-point-open topology ph is denoted by $C_{ph}(X)$. One can also view the bi-point-open topology on $C(X)$ as the weak topology on $C(X)$ generated by the identity maps $id_1 : C(X) \mapsto C_p(X)$ and $id_2 : C(X) \mapsto C_h(X)$.

In [3] and [2], the separation and countability properties of these two topologies on $C(X)$ have been studied.

In [3] the following statements were proved.

- $C_h(\mathbb{P})$ is separable where \mathbb{P} is the set of irrational numbers. (Proposition 5.1.)
- If $C_h(X)$ is separable, then every open subset of X is uncountable. (Theorem 5.2.)
- If X has a countable π -base consisting of nontrivial connected sets, then $C_h(X)$ is separable. (Theorem 5.5.)
- If $C_{ph}(X)$ is separable, then every open subset of X is uncountable. (Theorem 5.8.)
- If X has a countable π -base consisting of nontrivial connected sets and a coarser metrizable topology, then $C_{ph}(X)$ is separable. (Theorem 5.10.)

In [1] was definition a \mathcal{I} -set and was proved that necessary condition for $C_h(X)$ be a separable space is condition: X has a π -network consisting of \mathcal{I} -sets.

- A set $A \subseteq X$ be called \mathcal{I} -set if there is a continuous function $f \in C(X)$ such that $f(A)$ contains an interval $\mathcal{I} = [a, b] \subset \mathbb{R}$.
- If $C_h(X)$ is separable space, then X has a π -network consisting of \mathcal{I} -sets. (Theorem 2.3.)

In this paper we use the following conventions. The symbols \mathbb{R} , \mathbb{P} , \mathbb{Q} and \mathbb{N} denote the space of real numbers, irrational numbers, rational numbers and natural numbers, respectively. Recall that a dispersion character $\Delta(X)$ of X is the minimum of cardinalities of its nonempty open subsets.

Recall also that a space be called Polish space if it is a separable complete metrizable space.

By *set of reals* we mean a zero-dimensional, separable metrizable space every non-empty open set which has the cardinality the continuum.

2. Main results

Note that if the space $C_h(X)$ is a separable space then $\Delta(X) \geq \mathfrak{c}$. Really, if $A = \{f_i\}$ is a countable dense set of $C_h(X)$ then for each non-empty open set U of X we have $\bigcup f_i(U) = \mathbb{R}$. It follows that $|U| \geq \mathfrak{c}$.

Also note that if the space $C_{ph}(X)$ is a separable space then $C_p(X)$ is a separable space and $C_h(X)$ is a separable. It follows that X is a separable submetrizable (coarser separable metric topology) space and $\Delta(X) = \mathfrak{c}$.

Recall that a family γ of subsets of a space X is called T_0 -separating if whenever x and y are distinct points of X , there exists $V \in \gamma$ containing exactly one of the points x and y .

In [1] was proved the following result.

Theorem 2.1. *If X is a Tychonoff space with network consisting non-trivial connected sets, then the following are equivalent.*

1. $C_{ph}(X)$ is a separable space.
2. X is a separable submetrizable space.

In the present paper, we consider more wider form this theorem.

Theorem 2.2. *If X is a Tychonoff space with π -network consisting non-trivial connected sets, then the following are equivalent.*

1. $C_{ph}(X)$ is a separable space.
2. X has a countable T_0 -separating family of zero-sets.
3. X is a separable submetrizable space.

Proof. (1) \Rightarrow (2). Let $C_{ph}(X)$ be separable space. There is a countable dense subset $A = \{f_i\}$ of the space $C_{ph}(X)$. Fix $\beta = \{B_j\}$ some countable base for \mathbb{R} consisting of bounded open intervals.

Consider $\gamma = \{f^{-1}(\overline{B}) : f \in A, B \in \beta\}$.

We show that a countable family γ is required family.

Let x_1 and x_2 be distinct points of X . Consider open base set $Q = [\{x_1\}, (c_1, d_1)]^+ \cap [\{x_2\}, (c_2, d_2)]^+$ of the space $C_{ph}(X)$ where $(c_i, d_i) \in \beta$ for $i = 1, 2$ and $\overline{(c_1, d_1)} \cap \overline{(c_2, d_2)} = \emptyset$. There is $h \in Q \cap A$. Clearly that $h^{-1}(\overline{(c_i, d_i)}) \in \gamma$, $x_i \in h^{-1}(\overline{(c_i, d_i)})$ for $i = 1, 2$ and $h^{-1}(\overline{(c_1, d_1)}) \cap h^{-1}(\overline{(c_2, d_2)}) = \emptyset$.

The countable family γ is required family.

(2) \Rightarrow (3). $\gamma = \{Z_i\}$ be the countable family with required conditions. We can assume that γ is closed under finite unions.

For each $i \in \mathbb{N}$ there is continuous function $f_i : X \mapsto I = [0, 1]$ such that $Z_i = f_i^{-1}(0)$.

Let $I_i = I \times \{i\}$ for every $i \in \mathbb{N}$.

By letting

$(x, i_1)E(y, i_2)$ whenever $x = 0 = y$ or $x = y$ and $i_1 = i_2$

we define an equivalence relation E on the set $\bigcup_{i \in \mathbb{N}} I_i$.

The formula

$$\rho([(x, i_1)], [(y, i_2)]) = \begin{cases} |x - y|, & \text{if } i_1 = i_2, \\ x + y, & \text{if } i_1 \neq i_2, \end{cases}$$

defines a metric on the set of equivalence classes of E . This space - as well as the corresponding metrizable space - be called the *metrizable hedgehog of spininess* \aleph_0 and be denoted $J(\omega)$ (Example 4.1.5 in [4]).

Note that for every $i \in \mathbb{N}$ the mapping j_i of the interval I to $J(\omega)$ defined by letting $j_i(x) = [(x, i)]$ is a homeomorphic embedding. The family of all balls with rational radii around points of the form $[(r, i)]$, where r is a rational number, is a base for $J(\omega)$; so that $J(\omega)$ is a separable metrizable space.

The formula $h_i(x) = j_i(f_i(x))$ defines a continuous mapping $h_i : X \mapsto J(\omega)$. Note that the family $\{h_i\}_{i=1}^\infty$ is functionally separates points of X . Really let x and y be distinct points of X . There exists $Z \in \gamma$ containing exactly one of the points x and y and there are $i' \in \mathbb{N}$ and continuous function $f_{i'} : X \mapsto I = [0, 1]$ such that $Z = f_{i'}^{-1}(0)$. Hence $h_{i'}(x) \neq h_{i'}(y)$. Thus diagonal mapping $h = \bigtriangleup_{i \in \mathbb{N}} h_i : X \mapsto J(\omega)^\omega$ is a continuous one-to-one mapping from X into the separable metrizable space $J(\omega)^\omega$.

It follows that X is a separable submetrizable space.

(3) \Rightarrow (1). Let X be a separable submetrizable space, i.e. X has coarser separable metric topology τ_1 and γ be π -network of X consisting non-trivial connected sets. Let $\beta = \{B_i\}$ be a countable base of (X, τ_1) . We can assume that β closed under finite union of its elements.

For each finite family $\{B_{s_i}\}_{i=1}^d \subset \beta$ such that $\overline{B_{s_i}} \cap \overline{B_{s_j}} = \emptyset$ for $i \neq j$ and $i, j \in \overline{1, d}$ and $\{p_i\}_{i=1}^d \subset \mathbb{Q}$ we fix $f = f_{s_1, \dots, s_d, p_1, \dots, p_d} \in C(X)$ such that $f(\overline{B_{s_i}}) = p_i$ for each $i = \overline{1, d}$.

Let G be the set of functions $f_{s_1, \dots, s_d, p_1, \dots, p_d}$ where $s_i \in \mathbb{N}$ and $p_i \in \mathbb{Q}$ for $i \in \mathbb{N}$. We claim that the countable set G is dense set of $C_{ph}(X)$.

By proposition 2.2 in [3], let

$W = [x_1, V_1]^+ \cap \dots \cap [x_m, V_m]^+ \cap [U_1, r_1]^- \cap \dots \cap [U_n, r_n]^-$ be a base set of $C_{ph}(X)$ where $n, m \in \mathbb{N}$, $x_i \in X$, V_i is open set of \mathbb{R} for $i \in \overline{1, m}$, U_j is open set of X and $r_j \in \mathbb{R}$ for $j \in \overline{1, n}$ and for $i \neq j$, $x_i \neq x_j$ and $\overline{U_i} \cap \overline{U_j} = \emptyset$.

Choose $B_{s_l} \in \beta$ for $l = \overline{1, m+n}$ such that

1. $\overline{B_{s_{l_1}}} \cap \overline{B_{s_{l_2}}} = \emptyset$ for $l_1 \neq l_2$ and $l_1, l_2 \in \overline{1, n+m}$;
2. $x_i \in B_{s_l}$ for $l \in \overline{1, m}$;
3. $B_{s_l} \cap U_k \neq \emptyset$ for $l \in \overline{m+1, n+m}$ and $k = l - m$.

Choose $B_{s'_l} \in \beta$ for $l \in \overline{1, m}$ such that $x_i \in B_{s'_l}$ and $\overline{B_{s'_l}} \subseteq B_{s_l}$.

Choose $A_k \in \gamma$ for $k \in \overline{1, n}$ such that $A_k \subseteq (U_k \cap \overline{B_{s_l}})$ where $l = k + m$.

Choose different points $s_k, t_k \in A_k$ for every $k \in \overline{1, m}$.

Let $S, T \in \beta$ such that $\overline{S} \cap \overline{T} = \emptyset$, $\overline{B_l} \cap \overline{S} = \emptyset$, $\overline{B_l} \cap \overline{T} = \emptyset$ for $l \in \overline{1, m}$ and $s_k \in S$ and $t_k \in T$ for all $k \in \overline{1, m}$.

Fix points $v_i \in (V_i \cap \mathbb{Q})$ for $i \in \overline{1, m}$.

Choose $p, q \in \mathbb{Q}$ such that $p < \min\{r_i : i \in \overline{1, n}\}$ and $q > \max\{r_i : i \in \overline{1, n}\}$.

Let

$$f(x) = \begin{cases} p & \text{for } x \in \overline{S} \\ q & \text{for } x \in \overline{T} \\ v_l & \text{for } x \in \overline{B_{s'_l}} \end{cases}$$

where $l \in \overline{1, m}$.

Note that $f \in W \cap G$. This proves theorem. □

In [1] the following statements were proved.

Theorem 2.3. *Let X be a Tychonoff space with countable π -base, then the following are equivalent.*

1. $C_{ph}(X)$ is a separable space.
2. X is separable submetrizable space and it has a countable π -network consisting of \mathcal{I} -sets.

Theorem 2.4. *Let X be a Tychonoff space with countable π -base, then the following are equivalent.*

1. $C_h(X)$ is a separable space.
2. X has a countable π -network consisting of \mathcal{I} -sets.

The next result is corollary of Theorem 2.3, but we notes its as theorem due to the importance of the class of separable metrizable spaces.

Theorem 2.5. *If X is a separable metrizable space, then the following are equivalent.*

1. $C_{ph}(X)$ is a separable space.
2. X has a countable π -network consisting of \mathcal{I} -sets.

We have already noted that if the space $C_{ph}(X)$ is a separable space then

- X is a separable submetrizable;
- X has a π -network consisting of \mathcal{I} -sets.

Theorem 2.6. *If X is a separable submetrizable space with countable π -network consisting of \mathcal{I} -sets, then $C_{ph}(X)$ is separable space.*

Proof. The proof analogously to the proof of the implication $((2) \Rightarrow (1))$ in Theorem 2.4 ([1]).

Let $S = \{S_i\}$ be a countable π -network of X consisting of \mathcal{I} -sets. By definition of \mathcal{I} -sets, for each $S_i \in S$ there is the continuous function $h_i \in C(X)$ such that $h_i(S_i)$ contains an interval $[a_i, b_i]$ of real line. Consider a countable set

$$\{h_{i,p,q}(x) = \frac{p-q}{a_i-b_i} * h_i(x) + p - \frac{p-q}{a_i-b_i} * a_i\}$$

of continuous functions on X , where $i \in \mathbb{N}$, $p, q \in \mathbb{Q}$. Note that if $h_i(x) = a_i$ then $h_{i,p,q}(x) = p$ and if $h_i(x) = b_i$ then $h_{i,p,q}(x) = q$.

Let $\beta = \{B_j\}$ be countable base of (X, τ_1) where τ_1 is separable metraizable topology on X because of X is separable submetrizable space. For each pair (B_j, B_k) such that $\overline{B_j} \subseteq B_k$ define continuous functions

$$h_{i,p,q,j,k}(x) = \begin{cases} h_{i,p,q}(x) & \text{for } x \in B_j \\ \mathbf{0} & \text{for } x \in X \setminus B_k. \end{cases}$$

and for each $v \in \mathbb{Q}$

$$d_{j,k,v}(x) = \begin{cases} v & \text{for } x \in B_j \\ \mathbf{0} & \text{for } x \in X \setminus B_k. \end{cases}$$

Let G be the set of finite sum of functions $h_{i,p,q,j,k}$ and $d_{j,k,v}$ where $i, j, k \in \mathbb{N}$ and $p, q, v \in \mathbb{Q}$. We claim that the countable set G is dense set of $C_{ph}(X)$.

By proposition 2.2 in [3], let

$W = [x_1, V_1]^+ \cap \dots \cap [x_m, V_m]^+ \cap [U_1, r_1]^- \cap \dots \cap [U_n, r_n]^-$ be a base set of $C_{ph}(X)$ where $n, m \in \mathbb{N}$, $x_i \in X$, V_i is open set of \mathbb{R} for $i \in \overline{1, m}$, U_j is open set of X and $r_j \in \mathbb{R}$ for $j \in \overline{1, n}$ and for $i \neq j$, $x_i \neq x_j$ and $\overline{U_i} \cap \overline{U_j} = \emptyset$.

Fix points $y_j \in U_j$ for $j = \overline{1, n}$ and choose $B_{s_l} \in \beta$ for $l = \overline{1, n+m}$ such that $\overline{B_{s_{l_1}}} \cap \overline{B_{s_{l_2}}} = \emptyset$ for $l_1 \neq l_2$ and $l_1, l_2 \in \overline{1, n+m}$ and $x_i \in B_{s_l}$ for $l \in \overline{1, m}$ and $y_j \in B_{s_l}$ for $l \in \overline{m+1, n}$. Choose $B_{s'_l} \in \beta$ for $l \in \overline{1, m}$ such that $x_i \in B_{s'_l}$ and $\overline{B_{s'_l}} \subseteq B_{s_l}$ and choose $B_{s'_l} \in \beta$ for $l \in \overline{m+1, n+m}$ such that $y_j \in \overline{B_{s'_l}} \subseteq B_{s_l}$ where $l = j + m$.

Fix points $v_i \in (V_i \cap \mathbb{Q})$ for $i \in \overline{1, m}$ and $p_j, q_j \in \mathbb{Q}$ such that $p_j < r_j < q_j$ for $j = \overline{1, n}$.

Consider $g \in G$ such that

$$g = d_{s'_1, s_1, v_1} + \dots + d_{s'_m, s_m, v_m} + h_{i_1, p_1, q_1, s'_{m+1}, s_{m+1}} + \dots + h_{i_n, p_n, q_n, s'_{m+n}, s_{m+n}}$$

where $S_{i_k} \subset B_{s'_l} \cap U_k$ for $k = \overline{1, n}$ and $l = k + m$.

Note that $g \in W \cap G$. This proves theorem. \square

Corollary 2.7. If X is a perfect Polish space, then $C_{ph}(X)$ is separable.

Proof. It follows immediately from fact that any regular closed subset of a space X is a perfect Polish space and it contains some set which is homeomorphic to 2^ω ([5]). It follows that any non-empty open set of X is \mathcal{I} -set. \square

Note that there is the example (Example 4.3. in [1]) such that

- X hasn't countable chain condition, hence, X hasn't countable π -network consisting of \mathcal{I} -sets;
- X is separable submetrizable space;
- $C_{ph}(X)$ is separable space.

Example 2.8. Let $X = \oplus_{\alpha < \mathfrak{c}} \mathbb{R}_\alpha$ be a direct sum of real lines \mathbb{R} .

In this connection a natural question arises.

Question 1. Let X be a separable submetrizable space with uncountable π -network consisting of \mathcal{I} -sets. Is $C_{ph}(X)$ separable?

Recall that a set of reals X is *null* if for each positive ϵ there exists a cover $\{I_n\}_{n \in \mathbb{N}}$ of X such that $\sum_n \text{diam}(I_n) < \epsilon$. A set of reals X has *strong measure zero* if, for each sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ of positive reals, there exists a cover $\{I_n\}_{n \in \mathbb{N}}$ of X such that $\text{diam}(I_n) < \epsilon_n$ for all n . For example, every Lusin set has strong measure zero.

In [1] (Example 3.1) was shown that it is consistent with ZFC that exists the separable metrizable space X such that $\Delta(X) = \mathfrak{c}$ and $C_{ph}(X)$ isn't separable.

Example 2.9. (CH) *Let X be a set of reals and it has strong measure zero.*

In [6] was shown that it is consistent with ZFC that for any set of reals of cardinality the continuum, there is a (uniformly) continuous map from that set onto the closed unit interval. In fact, this holds in the iterated perfect set model.

In [1] the following statement was proved.

Theorem 2.10. *(the iterated perfect set model)*

If X is a separable metrizable space, then the following are equivalent.

1. $C_{ph}(X)$ is a separable space.
2. $\Delta(X) = \mathfrak{c}$.

In the present paper, we consider more wider form this theorem.

Theorem 2.11. *(the iterated perfect set model) If X is a regular space with a countable network, then the following are equivalent.*

1. $C_{ph}(X)$ is a separable space.
2. $\Delta(X) = \mathfrak{c}$.
3. X has a countable π -network consisting of \mathcal{I} -sets.

Proof. (1) \Rightarrow (2). Note that if the space $C_{ph}(X)$ is a separable space then $C_p(X)$ is a separable space and $C_h(X)$ is a separable. It follows that X is a separable submetrizable space and, hence, $\Delta(X) = \mathfrak{c}$.

(2) \Rightarrow (3). Let $\Delta(X) = \mathfrak{c}$.

(I). We show that any separable metrizable space M of cardinality \mathfrak{c} is \mathcal{I} -set of M , i.e. there exist a continuous function $f : M \mapsto \mathbb{R}$ such that $f(M) \supseteq \mathcal{I}$.

Really, if a real-valued continuous image of space M has cardinality less \mathfrak{c} for any $f \in C(M)$, then M is zero-dimensional space. It follows that M is set of reals and, by the iterated perfect set model, there is a continuous map from that set onto the closed unit interval \mathcal{I} .

If there is real-valued continuous image of space M such that it has cardinality \mathfrak{c} , then either it contains an interval \mathcal{I} or it is set of reals and, again,

by the iterated perfect set model, there is a continuous map from that set onto the closed unit interval \mathcal{I} .

(II). Recall that a regular space with a countable network is normal and separable submetrizable space. Since $\Delta(X) = \mathfrak{c}$ and X is regular space with a countable network, it follows that X has countable π -network α consisting of closed subsets of cardinality \mathfrak{c} of X .

We show that α is required π -network.

Let f be a condensation from X onto a separable metrizable space. Fix $A \in \alpha$ and consider a mapping $h = f \upharpoonright A$. By point (I), $h(A)$ is \mathcal{I} -set of $h(A)$, i.e. there exist a continuous function $f : h(A) \mapsto \mathbb{R}$ such that $f(h(A)) \supseteq \mathcal{I}$. Since X is normal space, by Tietze-Urysohn Extension Theorem, the map $f \circ h$ can be extended to a real-valued continuous map $F : X \mapsto \mathbb{R}$. Note that $F(A) = f(h(A)) \supseteq \mathcal{I}$ i.e. A is \mathcal{I} -set of X .

(2) \Rightarrow (3). It follows from Theorem 2.6.

□

3. Acknowledgement

This work was supported by Act 211 Government of the Russian Federation, contract 02.A03.21.0006.

References

- [1] Alexander V. Osipov, *On separability of the functional space with the open-point and bi-point-open topologies*, arXiv:1602.02374v2[math.GN] 12 Feb 2016.
- [2] Anubha Jindal, R.A. McCoy, S. Kundu, *The open-point and bi-point-open topologies on $C(X)$: Submetrizability and cardinal functions*, Topology and its Applications, 196, (2015), p.229–240.
- [3] Anubha Jindal, R.A. McCoy, S. Kundu, *The open-point and bi-point-open topologies on $C(X)$* , Topology and its Applications, 187, (2015), p.62–74.
- [4] R. Engelking, *General Topology*, Revised and completed ed. - Berlin: Heldermann, (1989).
- [5] Kazimirz Kuratowski, *Topology*, Academic Press, Vol.I, (1966).

- [6] Arnold W. Miller, *Mapping a Set of Reals Onto the Reals*, Journal of Symbolic Logic, Vol. 48, Issue 3, (Sep.,1983), 575—584.